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INCOMPRESSIBLE NON-NEWTONIAN FLUIDS

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(NASA-CR-185780) ON THE NORMAL STRESS EFFECTS OF INCOMPRESSIBLE NON-NEWTONIAN FLUIDS (ICASE) 16 p

N90-70037

Unclas

00/34 0224391

Report No. 80-31

November 14, 1980

INSTITUTE FOR COMPUTER APPLICATIONS IN SCIENCE AND ENGINEERING

NASA Langley Research Center, Hampton, Virginia

Operated by the

UNIVERSITIES SPACE USRA RESEARCH ASSOCIATION

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INCOMPRESSIBLE NON-NEWTONIAN FLUIDS

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ABSTRACT

There are various viscometric flow models available at present to determine the mechanical properties of incompressible non-Newtonian fluids. In most of these measurements of the normal stresses is quite a challenge. We introduce a new viscometric flow model and derive a theory for the measurement of the "second normal stress." We also indicate how such a measurement can be performed in practice.

This research was supported by NASA Contract No. NAS1-15810 while the author was in residence at ICASE, NASA Langley Research Center, Hampton, VA 23665.

1. Introduction

At present there are various viscometric flow models (this terminology is due to Coleman [2]) available for incompressible now Newtonian fluids. This class of flows describes mechanical behavior of an incompressible fluid completely by three material functions η, σ_1 and σ_2 which depend on the rate of shear K. In the literature η is known as the viscosity function and σ_1 and σ_2 are known as the first and second normal stresses of the fluid. Couette flow and Poiseuille flow are well known examples of such flows. The fluid, in fact, can be Newtonian or non-Newtonian. In the case of Newtonian fluids, as we shall see later, the normal stresses σ_1 and σ_2 are zero. Viscometric flows are of great interest to rheologists.

The purpose of this note is to present a new viscometric flow model which has not been studied for non-Newtonian fluids. Also we introduce a theory for the calculation of the second normal stress σ_2 and present a scheme for measuring σ_2 in practice.

We shall begin by stating the fluid flow model. The flow is generated by dripping fluid on the top center of a vertical cylindrical block of radius R and of height h at a slow steady rate forming a thin layer on the walls (see Fig. 1).

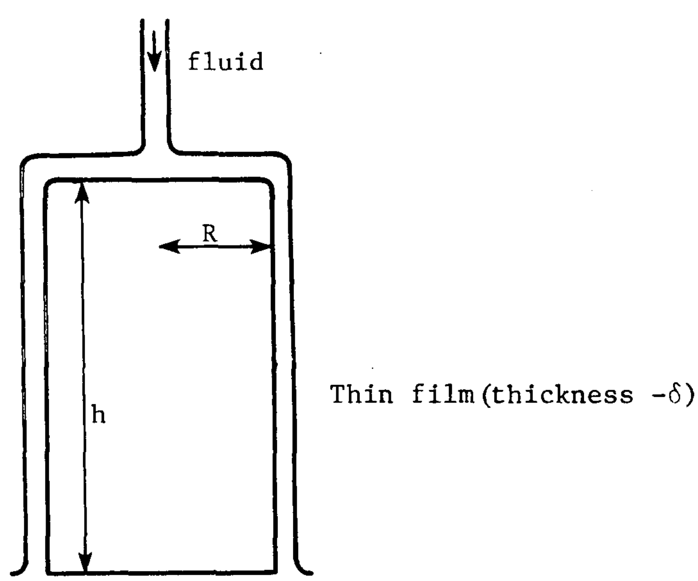

Figure 1

We assume that the thickness of the layer δ is constant throughout the motion. Discussion with a group of Polymer scientists indicates a difficulty with this assumption; that is, the thickness of the layer would be larger or smaller at the ends of the cylinder depending on the "thickening" or "thinning" property of the fluid. We shall overcome the difficulty by making the assumption that $h >> R$ so that the flow will have straight stream lines over a large area of the wall of the cylinder. We now discuss the equations governing this motion.

2. Equations of motion

Notation: let u denote the velocity of the fluid, T_e denote the extra stress and p denote the pressure field.

Then the momentum equation is

$$\rho \dot{u} = -\nabla p + \operatorname{div} T_e + \nabla (gz) , \tag{M}$$

and the incompressibility condition yields

$$\operatorname{div} \underline{u} = 0. \tag{I}$$

In equation (M) g is the acceleration due to gravity and gz provides the potential of the body force at a distance z measured from the top of the cyclinder. We assign a polar coordinate system from the top of the cyliner with the z axis pointing downwards. (See Fig. 2.)

In what follows we make the usual assumptions to simplify our mathematical model. These assumptions are more physical and rather straight forward. First we assume that the fluid velocity is in the z direction:

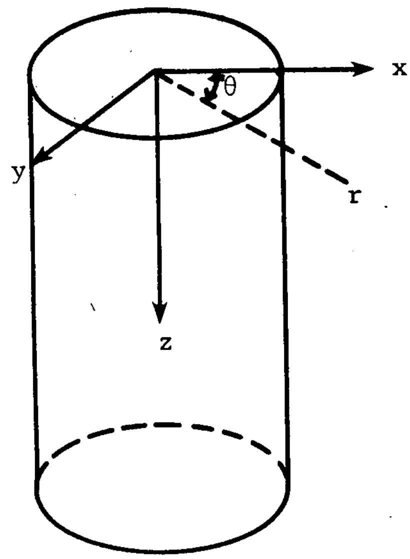

Figure 2.

$$u = u_z \hat{i}_z.$$

Then the incompressibility condition (I) becomes

$$\frac{\partial u_z}{\partial z} = 0.$$

This yields

$$u = u_z(r,\theta)\hat{i}_z.$$

Furthermore, we assume symmetry in the θ-direction so that u will have the form

$$u = u_z(r)\hat{i}_z.$$

For the steady flow problem, the assumption that streamlines are straight and parallel to the z-direction means that $\rho\dot{u}$ in equation (M) is zero; hence equation (M) reduces to

$$0 = -\nabla p + \operatorname{div} T_e + \nabla(gz). \tag{M'}$$

It is well known (Coleman and Noll [1]) that for steady viscometric flows the extra stress tensor T_e has the form

$$T_e = \begin{bmatrix} T_{rr} & T_{rz} & 0 \\ T_{zr} & T_{zz} & 0 \\ 0 & 0 & T_{\theta\theta} \end{bmatrix}, \tag{1}$$

where

$$T_{rz} = T_{zr} = \tau(K), \tag{2}$$

is known as the shear stress and K is the shear rate defined by

$$K = \frac{du_z}{dr}. \tag{3}$$

T_{rr}, T_{zz} and $T_{\theta\theta}$ are defined by

$$\begin{aligned} T_{rr} - T_{zz} &= \sigma_1(K), \\ T_{rr} - T_{\theta\theta} &= \sigma_2(K). \end{aligned} \tag{4}$$

The quantities $\sigma_1(K)$ and $\sigma_2(K)$ are known as the first and second normal stress of the fluid. This convention of representing σ_1 and σ_2 as the differences of normal stress components is similar to the one used in [4]. Various other authors use different conventions. For example Coleman and Noll [1] use the convention

$$\sigma_1(K) = T_{rr} - T_{zz} \quad ,$$

$$\sigma_2(K) = T_{\theta\theta} - T_{zz} \quad .$$

The shear dependent viscosity $\eta \equiv \eta(K)$ is identified with $\tau(K)$ in the rheological literature as

$$\tau(K) = K\eta(K).$$

For imcompressible Newtonian fluids

$$\tau(K) = K\eta_0, \qquad \eta_0 = \text{constant} > 0,$$

and for ideal or perfect fluids

$$\tau(K) \equiv \sigma_1(K) \equiv \sigma_2(K) \equiv 0.$$

For Reiner-Rivlin fluids $\sigma_1(K) \equiv \sigma_2(K)$ (see [3]). The function $\tau(K)$ is usually measured by a simple plane shearflow. Thus there are constitutive models for $\tau(K)$ for different fluids available in the literature. Also there are various constitutive models for $\sigma_1(K)$ in the polymer science literature. However, there are few models for $\sigma_2(K)$ because of experimental difficulties. This note presents a scheme for the measurement of σ_2. Also the model can be used as a visometer, that is, it can be used to measure the viscosity of Newtonian fluids. We shall discuss this in the next section under the calculation of the flow rate.

We observe from (3) that K is a function of r only and hence so is T_e. Under these conditions (M') can be written as

$$\frac{1}{r}\frac{d}{dr}(r\,T_{rr}) - \frac{T_{\theta\theta}}{r} = \frac{\partial p}{\partial r}\ , \tag{5}$$

$$0 = \frac{1}{r}\frac{\partial p}{\partial \theta}\ , \tag{6}$$

$$\frac{1}{r}\frac{d}{dr}(r\,T_{rz}) = \frac{\partial p}{\partial z} - g. \tag{7}$$

Equations (5) - (7) determine p in the form

$$p = a\,z + f(r), \tag{8}$$

where a, a constant, and f a function of r only, are to be determined.

The constant a in (8) is determined from the continuity of pressure across the free boundary at $r = R + \delta$,

$$p = a\,z + f(R+\delta) = p_0\ ,$$

for all values of z, where p_0 is the atmospheric pressure. Hence $a = 0$; and $p = f(r)$. In turn equation (7) gives us

$$\frac{1}{r}\frac{d}{dr}(r\ T_{rz}) = -g\ ,$$

or

$$T_{rz} = \tau(K(r)) = -g\,\frac{r}{2} + \frac{C}{r}\ ,$$

where C is a constant and we determine C by the fact that on the free boundary $r = R + \delta$, $\tau(K(r+\delta))$ - the shear stress - is zero. This gives

$$\tau(K) = -\frac{g}{2}\left[r - \frac{r_0^2}{r}\right]\ , \tag{9}$$

where

$$r_0 = R + \delta.$$

For most fluids the constitutive relation for τ can be obtained as a function of K. Thus equation (9) can be inverted in terms of K. That is,

$$K(r) = \tau^{-1}\left[-\frac{gr}{2} + \frac{g}{2}\frac{r_0^2}{r}\right]. \tag{10}$$

But $\dfrac{du_z(r)}{dr} = K(r)$. Also using the fact that $u_z(R) = 0$, we have

$$u_z(r) = \int_R^r \tau^{-1}\left[-\frac{g\xi}{2} + \frac{g}{2}\frac{r_0^2}{\xi^2}\right]d\xi\ . \tag{11}$$

Thus the velocity profile is determined. It can be shown (see [1]) that the normal stresses T_{rr} and $T_{\theta\theta}$ can be obtained in terms of first and second normal stresses as follows:

$$T_{rr} = (1/3)[2\sigma_1(K) - \sigma_2(K)] \quad ,$$
$$(12)$$
$$T_{\theta\theta} = (1/3)[\sigma_1(K) + \sigma_2(K)] \quad .$$

These relations will be used in solving for $f(r)$ in order to find the pressure field from equation (1). In principle, if we know the constitutive equations for σ_1 and σ_2 as functions of K we could solve equation (5). Unfortunately, although several models for σ_1 exist σ_2 is rarely known. In section 4 we give a procedure for the determination of σ_2.

3. Flow Rate

It is customary in viscometric flows to calculate the flow rate at the exit of the flow model. This calculation is used to obtain τ in terms of the flow rate. However, measurements done in this way are subject to errors due to the swelling effects of non-Newtonian fluids; for further details see [3]. The model can be used to measure the viscosity of any Newtonian fluids as we now describe.

Let Q be the volume discharge of the fluid at the end of the cylinder per unit time. Then

$$Q = 2\pi \int_R^{r_0} u_z(r)\, r\, dr \; , \tag{13}$$

using the boundary condition $u_z(R) = 0$ and the shear rate $\dfrac{du_z(r)}{dr} = K(r)$ in the above expression and integrating by parts we have

$$Q = \pi u_z(r_0) r_0^2 - \pi \int_R^{r_0} K(r) r^2 dr.$$

Using equation (10) we have

$$Q = \pi u_z(r_0)r_0^2 - \pi\int_R^{r_0} \tau^{-1}\left[-\frac{gr}{2}+\frac{g}{2}\frac{r_0^L}{r}\right] r^2 dr.$$

Thus if Q and the velocity on the boundary are known, in principle τ can be inverted and obtained in terms of Q and $u_z(r_0)$. As in the case of other models unless the magnitude of swelling is known we can not use this calculation to measure τ. But for Newtonian fluids, we note that

$$\tau(K) = \eta_0 K = \eta_0 \frac{du_z}{dr} .$$

From equation (9) we have

$$\frac{du_z}{dr} = -\frac{g}{2\eta_0}\left[r - \frac{r_0^2}{r}\right].$$

Integration of the last equation using the boundary condition $u_z(R) = 0$ yields

$$u_z(r) = -\frac{g}{4\eta_0}\left[\tfrac{1}{2}(r^2 - R^2) - r_0^2 \log(\tfrac{r}{R})\right].$$

Substituting this value of u_z in equation (13) we have,

$$Q = -\frac{\pi g}{2\eta_0}\int_R^{r_0}\left[\tfrac{1}{2}(r^3 - R^2 r) - r_0^2 r \log(\tfrac{r}{R})\right]dr,$$

or

$$\eta_0 = \frac{\pi g}{8}\left[4r_0^4 \log\frac{r_0}{R} - (3r_0^2 - R^2)(r_0^2 - R^2)\right] ,$$

which gives the viscosity.

4. Theory for the Calculation of Second Normal Stress.

For convenience we rewrite equation (1) with $p = f(r)$:

$$\frac{d}{dr} T_{rr} + \frac{1}{r}(T_{rr} - T_{\theta\theta}) = f'(r). \qquad (14)$$

We observe that again

$$T_{rr}(r) - T_{\theta\theta}(r) = \sigma_2(K(r)).$$

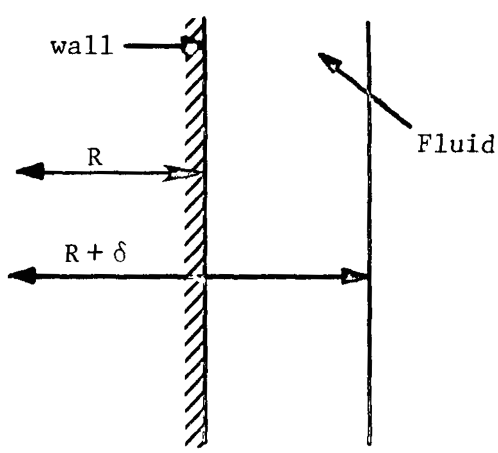

Figure 3

We integrate equation (14) over $[R+\delta, R]$ and let the atmospheric pressure be zero for convenience. The result is

$$\int_{R+\delta}^{R} \frac{dT_{rr}}{dr}\, dr + \int_{R+\delta}^{R} \frac{\sigma_2(K(\xi))}{\xi}\, d\xi = f(R).$$

Since the normal stress is zero on the free boundary $r = R + \delta$, we have $T_{rr}(R+\delta) = 0$. Thus the last equation simplifies to

$$T_{rr}(R) - f(R) = \int_{R}^{R+\delta} \frac{\sigma_2(K(\xi)d\xi}{\xi} \quad .$$

The quantity $T_{rr}(R)$ is the normal stress on the boundary in the radial direction and $f(R)$ is the pressure. This difference could be directly measured by a pitot tube as shown in Figure 4.

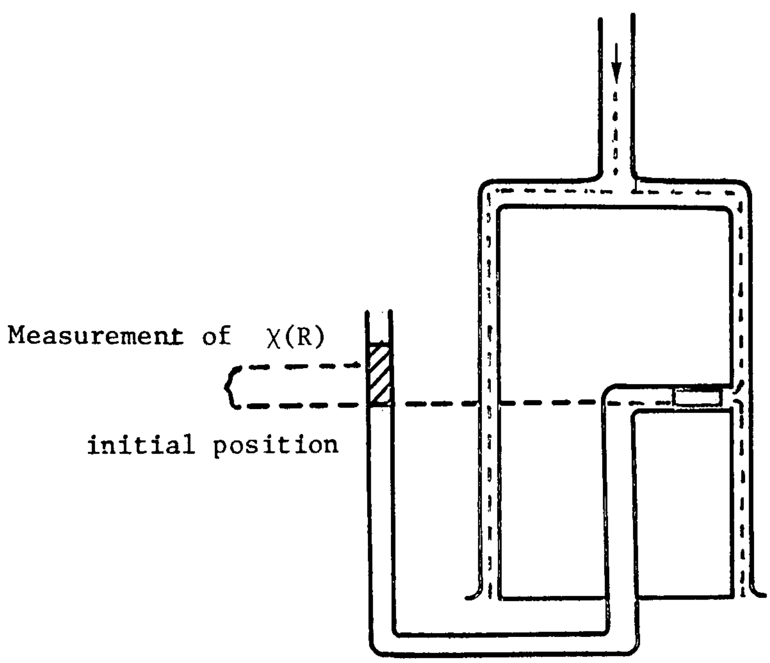

Figure 4.

Let

$$\chi(R) = T_{rr}(R) - f(R). \tag{15}$$

Thus

$$\chi(R) = \int_R^{R+\delta} \frac{\sigma_2(K(\xi))d\xi}{\xi} . \tag{16}$$

We will obtain an expression for σ_2 from equation (16). First we consider the expression (9) for $\tau(K)$. This can be rewritten as

$$\frac{g}{2} r^2 + r\tau - \frac{g}{2} r_0^2 = 0 \quad ,$$

which yields

$$r = - \frac{\tau \pm \sqrt{\tau^2 + (gr_0)^2}}{g} . \tag{17}$$

We observe that only the '+' sign is permissible. Also from equation (9) we have

$$d\tau = - \frac{g}{2} \left[\frac{r_0^2}{r^2} + 1 \right] dr \ ,$$

or

$$= - \frac{g}{2} \left[\frac{r_0^2}{r^2} + r \right] \frac{dr}{r} \ .$$

We use equation (17) to substitute for r in the brackets. The result is

$$d\tau = - \frac{g}{2} \frac{dr}{r} \left[\frac{gr_0^2\left(\tau + \sqrt{\tau^2 + (gr_0)^2}\right)}{g^2 r_0^2} - \frac{\left(\tau - \sqrt{\tau^2 + (gr_0)^2}\right)}{g} \right],$$

or

$$d\tau = -\sqrt{\tau^2+(gr_0)^2}\ \frac{dr}{r}\ .$$

Hence using this expression in equation (16) we have

$$\chi(R) = -\int_{\bar{\tau}}^{0} \frac{\sigma_2(K)}{\sqrt{\tau^2+(gr_0)^2}}\, d\tau\ ,$$

or

$$\chi(R) = \int_{0}^{\bar{\tau}} \frac{\sigma_2(K)}{\sqrt{\tau^2+(gr_0)^2}}\, d\tau\ , \qquad (18)$$

where $\bar{\tau}$ is the shear stress on the wall:

$$\bar{\tau} = \tau(K(R)).$$

Differentiating χ in (18) with respect to $\bar{\tau}$, we have

$$\frac{d\chi}{d\bar{\tau}} = \frac{\sigma_2(\bar{K})}{\sqrt{\bar{\tau}^2+(gr_0)^2}}\ , \qquad (19)$$

where $\bar{K}$ is the rate of shear on the wall. In principle $\frac{d\chi}{d\bar{\tau}}$ can be measured from experiments or can be calculated since χ and $\bar{\tau}$ are known. This analysis provides a means for measuring the second normal stress.

Acknowledgment

The author acknowledges useful discussions on this model with Dr. Y. Einaga and Mr. S. Venkatraman of the Department of Chemistry, Carnegie-Mellon University, Pittsburgh, PA. Also the author wishes to acknowledge the participation of Ms. Kathleen Spear of the Department of Mathematics in the early formulation of the problem.

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